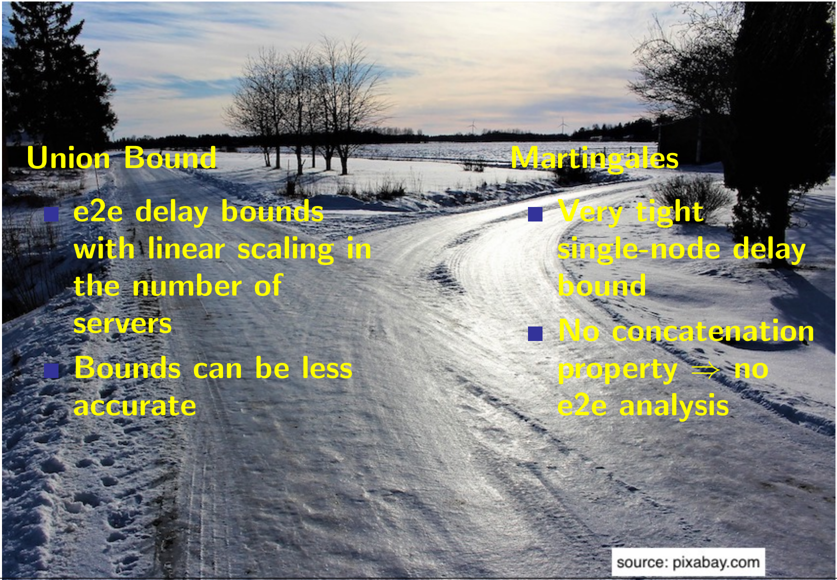
Improving Output Bounds in the Stochastic Network Calculus Using Lyapunov's Inequality

Paul Nikolaus and Jens Schmitt

WoNeCa 2018

Current State of SNC reaching a Crossroads



Union Bound

- e2e delay bounds with linear scaling in the number of servers
- Bounds can be less accurate

Martingales

- Very tight single-node delay bound
- No concatenation property $\Rightarrow$ no e2e analysis

source: pixabay.com

Current State of SNC reaching a Crossroads

Union Bound

- e2e delay bounds with linear scaling in the number of servers
- Bounds can be less accurate

Martingales

- Very tight single-node delay bound
- No concatenation property ⇒ no e2e analysis

Our goal: find a way to improve the "old" union bound

source: pixabay.com

MGF-Calculus

Theorem

The violation probability of a given stochastic delay bound T at time t is bounded by

$$\mathrm{P}(d(t) > T) \leq \mathrm{E}\left[e^{\theta(A \oslash S)(t+T,t)}\right]$$
$$= \mathrm{E}\left[\max_{0 \leq i \leq t} e^{\theta(A(i,t)-S(i,t+T))}\right]$$
$$\overset{\text{(UB)}}{\leq} \sum_{i=0}^{t} \mathrm{E}\left[e^{\theta(A(i,t)-S(i,t+T))}\right]$$

- Purpose of the union bound in the analysis: get rid of the max.

Union Bound

- does not need additional assumptions, independent of the distribution
- "is not going to be so bad if the variables (...) are rather uncorrelated" [Talagrand, 1996] or for Poisson arrivals [Ciucu, 2007]

Union Bound

- does not need additional assumptions, independent of the distribution
- "is not going to be so bad if the variables (...) are rather uncorrelated" [Talagrand, 1996] or for Poisson arrivals [Ciucu, 2007]
- is known to perform poorly for arrivals with correlated increments, such as Markov-Modulated On-Off traffic [Ciucu et al., 2013] [Beck, 2016]

Goal:
Mitigate the union bound's inaccuracy

$$\mathrm{E}\left[\max_{i=1,\ldots,n} e^{\theta X_i}\right] \leq \sum_{i=1}^{n} \mathrm{E}\left[e^{\theta X_i}\right]$$

but still obtain *end-to-end delay bounds*.

Lyapunov's Inequality

Proposition

Let $X \geq 0$ be in $\mathcal{L}^l$ with $l \geq 1$. Then it holds that

$$\mathrm{E}[X] \leq \left(\mathrm{E}\left[X^l\right]\right)^{\frac{1}{l}}.$$

Lyapunov's Inequality

Proposition

Let $X \geq 0$ be in $\mathcal{L}^l$ with $l \geq 1$. Then it holds that

$$\mathrm{E}[X] \leq \left(\mathrm{E}\left[X^l\right]\right)^{\frac{1}{l}}.$$

As $l = 1$ is feasible, this yields

$$\mathrm{E}[X] = \inf_{l \geq 1} \left\{ \left(\mathrm{E}\left[X^l\right]\right)^{\frac{1}{l}} \right\}.$$

Adding Lyapunov's Inequality to the Delay Bound

$$
\begin{aligned}
\mathrm{P}(d(t) > T) &\leq \mathrm{E}\left[e^{\theta (A \oslash S)(t+T,t)}\right] \\
&= \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,t+T)\}}\right]
\end{aligned}
$$

Adding Lyapunov's Inequality to the Delay Bound

$$
\begin{aligned}
P(d(t) > T) &\leq \qquad E\left[e^{\theta (A \oslash S)(t+T,t)}\right] \\
&= E\left[e^{\theta \max_{0 \leq i \leq t}\{A(i,t) - S(i,t+T)\}}\right] \\
&\overset{(UB)}{\leq} \sum_{i=0}^{t} E\left[e^{\theta (A(i,t) - S(i,t+T))}\right]
\end{aligned}
$$

Adding Lyapunov's Inequality to the Delay Bound

$$
\begin{aligned}
\mathrm{P}(d(t) > T) &\leq \mathrm{E}\left[e^{\theta(A \oslash S)(t+T,t)}\right] \\
&= \mathrm{E}\left[e^{\theta \max_{0 \leq i \leq t}\{A(i,t)-S(i,t+T)\}}\right] \\
&\overset{(\mathrm{LI})}{=} \inf_{l \geq 1} \left\{ \left(\mathrm{E}\left[e^{l\theta \max_{0 \leq i \leq t}\{A(i,t)-S(i,t+T)\}}\right]\right)^{\frac{1}{l}} \right\}
\end{aligned}
$$

Adding Lyapunov's Inequality to the Delay Bound

$$
\begin{aligned}
\mathrm{P}(d(t) > T) &\leq \mathrm{E}\left[e^{\theta(A \oslash S)(t+T,t)}\right] \\
&= \mathrm{E}\left[e^{\theta \max_{0 \leq i \leq t}\{A(i,t)-S(i,t+T)\}}\right] \\
&\overset{(\mathrm{LI})}{=} \inf_{l \geq 1}\left\{\left(\mathrm{E}\left[e^{l\theta \max_{0 \leq i \leq t}\{A(i,t)-S(i,t+T)\}}\right]\right)^{\frac{1}{l}}\right\} \\
&\overset{(\mathrm{UB})}{\leq} \inf_{l \geq 1}\left\{\left(\sum_{i=0}^{t} \mathrm{E}\left[e^{l\theta(A(i,t)-S(i,t+T))}\right]\right)^{\frac{1}{l}}\right\}
\end{aligned}
$$

Adding Lyapunov's Inequality to the Delay Bound

$$\begin{aligned}
\mathrm{P}(d(t) > T) &\leq \mathrm{E}\left[e^{\theta(A \oslash S)(t+T,t)}\right] \\
&= \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,t+T)\}}\right] \\
&\overset{(\mathrm{LI})}{=} \inf_{l\geq 1}\left\{\left(\mathrm{E}\left[e^{l\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,t+T)\}}\right]\right)^{\frac{1}{l}}\right\} \\
&\overset{(\mathrm{UB})}{\leq} \inf_{l\geq 1}\left\{\left(\sum_{i=0}^{t}\mathrm{E}\left[e^{l\theta(A(i,t)-S(i,t+T))}\right]\right)^{\frac{1}{l}}\right\} \\
&\vdots \\
&\leq \sum_{i=0}^{t}\mathrm{E}\left[e^{\theta(A(i,t)-S(i,t+T))}\right],
\end{aligned}$$

as $l = 1$ is feasible.

Significant Improvement, but wait a Minute...

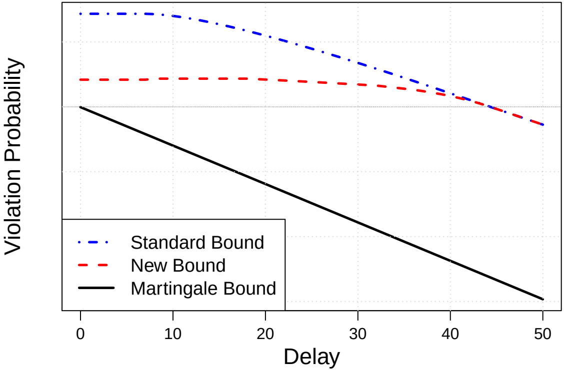


Figure: Single node delay bound with MMOO arrivals and constant rate server

Lyapu-Fail?

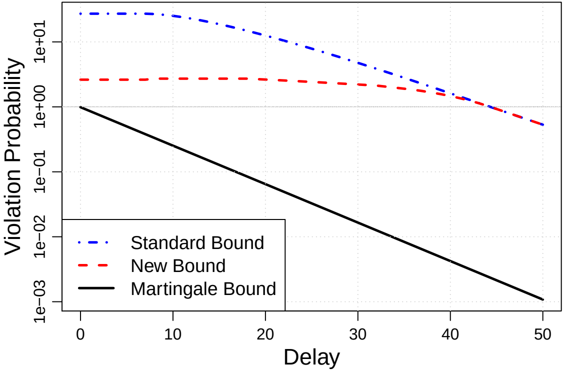


Figure: Single node delay bound with MMOO arrivals and constant rate server

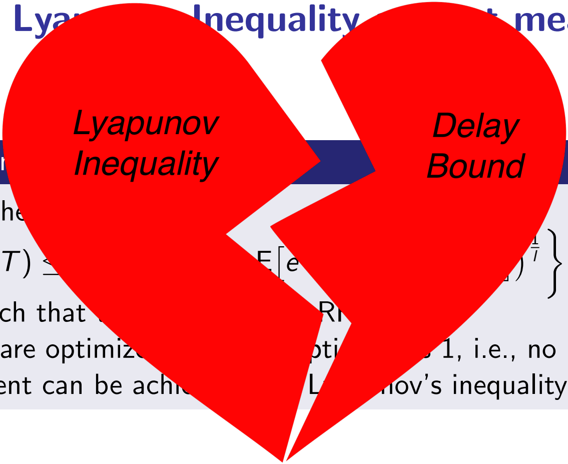
- Numerical experiments revealed that no improvement at all is achieved for bounds < 1!

Delay and Lyapunov Inequality are not meant to be together!

Proposition

Let T in the delay bound
$\mathrm{P}(d(t) > T) \leq \inf_{l \geq 1} \left\{ \left(\sum_{i=0}^{t} \mathrm{E}\left[e^{l\theta(A(i,t) - S(i,t+T))} \right] \right)^{\frac{1}{l}} \right\}$ be large enough such that the bound on the RHS is < 1.
If l and θ are optimized, then the optimal l is 1, i.e., no improvement can be achieved with Lyapunov's inequality.

Delay and Lya … Inequality … meant to be together!



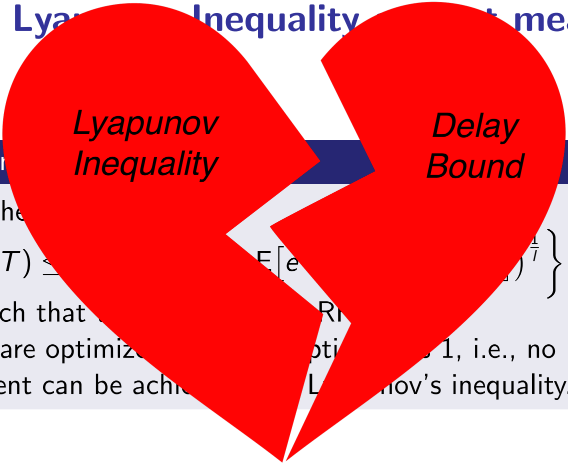
Propositio…

Let T in the …

$P(d(t) > T) \leq$ … $E\big[e$ … $\big)^{\frac{1}{l}}\Big\}$ be large

enough such that … R…
If l and θ are optimiz… …pti… 1, i.e., no
improvement can be achi… L…nov's inequality.

Delay and Lya Inequality t meant to be together!



Propositio

Let T in the

$\mathrm{P}(d(t) > T) \leq$ E[e) $^{\frac{1}{l}}$ } be large

enough such that R

If l and θ are optimiz pti 1, i.e., no

improvement can be achi L nov's inequality.

$\Rightarrow$ *Can it still be of any use?*

Improving the Output Bound

- Cannot improve probability bounds, such as $\mathrm{P}(d(t) > T)$
- Apply to MGF of output $\mathrm{E}\left[e^{\theta A'(s,t)}\right]$ instead

Improving the Output Bound

Let A' be the according output process. For the MGF bound we obtain

$$\mathrm{E}\left[e^{\theta A'(s,t)}\right] \leq \quad \mathrm{E}\left[e^{\theta \max_{0\leq i \leq t}\{A(i,t)-S(i,s)\}}\right]$$

Improving the Output Bound

Let A' be the according output process. For the MGF bound we obtain

$$\mathrm{E}\left[e^{\theta A'(s,t)}\right] \leq \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right]$$

$$\overset{\text{(UB)}}{\leq} \sum_{i=0}^{t} \mathrm{E}\left[e^{\theta(A(i,t)-S(i,s))}\right]$$

Improving the Output Bound

Let A' be the according output process. For the MGF bound we obtain

$$\mathrm{E}\left[e^{\theta A'(s,t)}\right] \leq \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right]$$

$$\overset{(\mathrm{LI})}{=} \inf_{l\geq 1}\left\{\left(\mathrm{E}\left[e^{l\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right]\right)^{\frac{1}{l}}\right\}$$

Improving the Output Bound

Let A' be the according output process. For the MGF bound we obtain

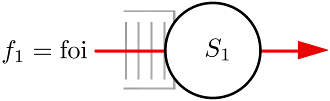
$$
\begin{aligned}
\mathrm{E}\left[e^{\theta A'(s,t)}\right] &\leq \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right] \\
&\overset{(\mathrm{LI})}{=} \inf_{l\geq 1}\left\{\left(\mathrm{E}\left[e^{l\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right]\right)^{\frac{1}{l}}\right\} \\
&\overset{(\mathrm{UB})}{\leq} \inf_{l\geq 1}\left\{\left(\sum_{i=0}^{t} \mathrm{E}\left[e^{l\theta(A(i,t)-S(i,s))}\right]\right)^{\frac{1}{l}}\right\}
\end{aligned}
$$

Improving the Output Bound

Let A' be the according output process. For the MGF bound we obtain

$$
\begin{aligned}
\mathrm{E}\left[e^{\theta A'(s,t)}\right] &\leq \mathrm{E}\left[e^{\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right] \\
&\overset{(LI)}{=} \inf_{l\geq 1}\left\{\left(\mathrm{E}\left[e^{l\theta \max_{0\leq i\leq t}\{A(i,t)-S(i,s)\}}\right]\right)^{\frac{1}{l}}\right\} \\
&\overset{(UB)}{\leq} \inf_{l\geq 1}\left\{\left(\sum_{i=0}^{t}\mathrm{E}\left[e^{l\theta(A(i,t)-S(i,s))}\right]\right)^{\frac{1}{l}}\right\} \\
&\quad\vdots \\
&\leq \sum_{i=0}^{t}\mathrm{E}\left[e^{\theta(A(i,t)-S(i,s))}\right].
\end{aligned}
$$

Improving the Output Bound



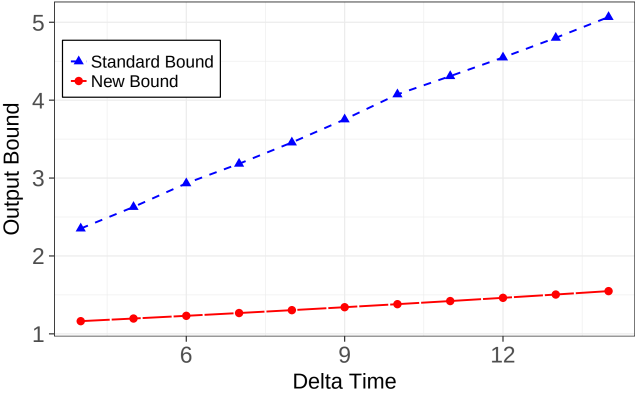
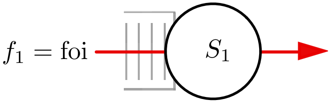


Figure: Output bound with MMOO arrivals and constant rate server.

Average Improvement is quite decent



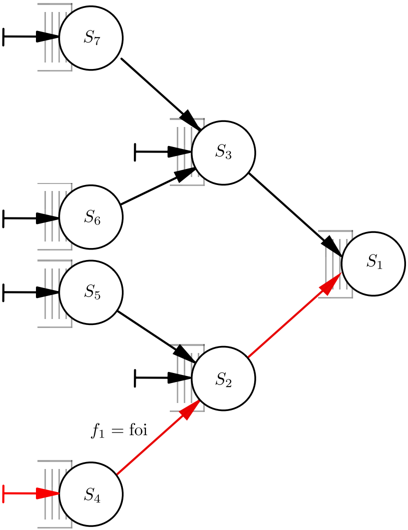
- Single node output with MMOO arrivals and constant rate server
- Randomly chosen parameters in a Monte-Carlo type fashion
- Investigate average and maximum improvement factor:

$$\frac{\text{Standard output bound}}{\text{New output bound}}$$

	Improvement Factor
Average	1.66
Maximum	400

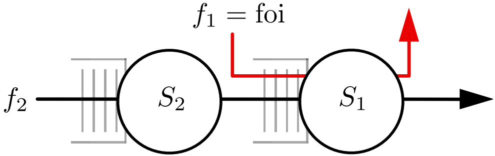
Hope for the Delay?

- Direct application to the delay bound is not possible
- In a larger network, multiple output bound computations are necessary to obtain the delay bound
- Need $2^h - (h+1)$ output bound computations to obtain delay bound on RHS



Application to canonical Topology

- Canonical setting that captures the effect on the delay bound:



Delay Bound can now be improved...

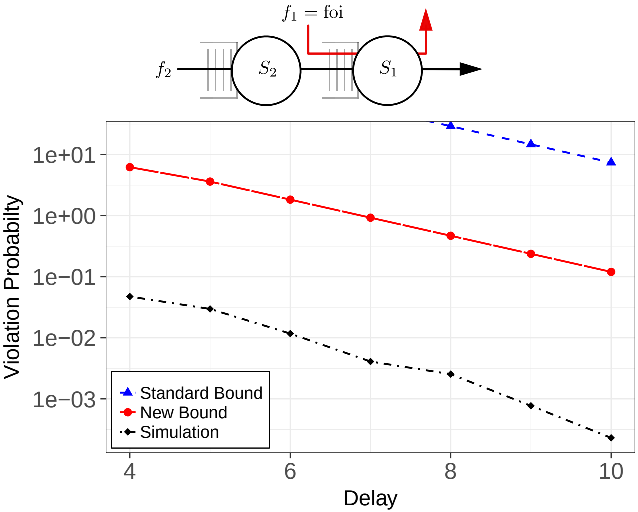
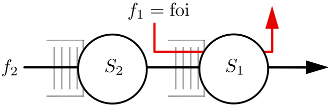


Figure: Delay bound with MMOO arrivals and constant rate server.

...but the Improvement is less significant on average



- As for the output bounds, evaluation by a Monte-Carlo type parameter space exploration
- Compute

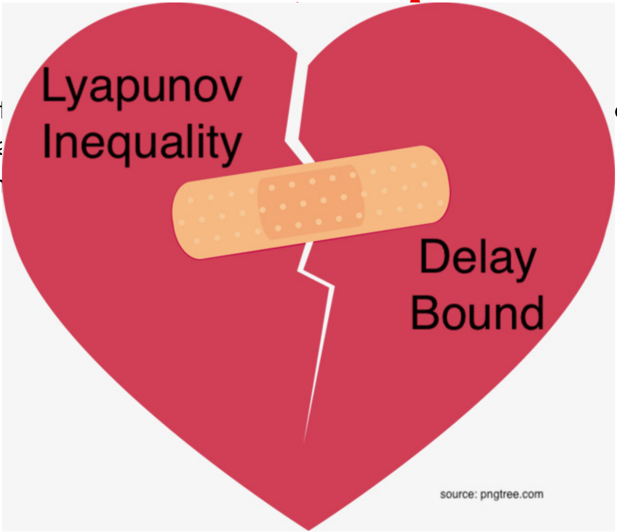
$$\frac{\text{Standard delay bound}}{\text{New delay bound}}$$

	Improvement Factor
Average	1.18
Maximum	323

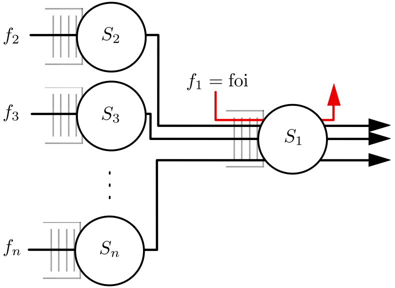
...but the Improvement is less significant on average

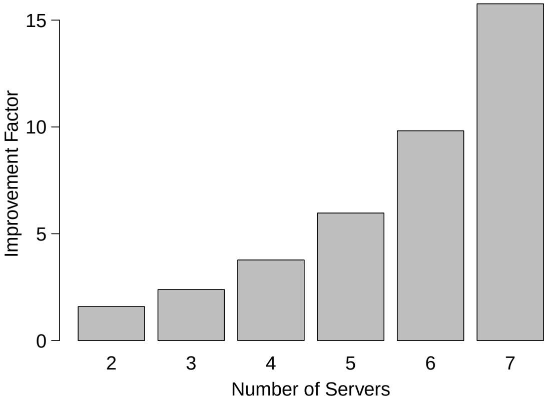
$f_1 = \text{foi}$

- As t... o type para...
- Con...



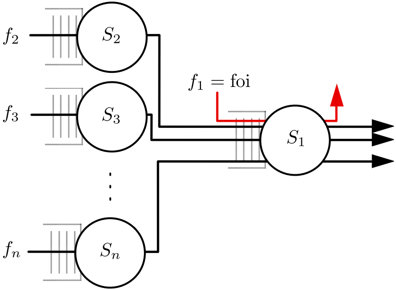
Improvement scales with the Number of Applications





- Lyapunov parameters $l_i, i = 1, \ldots, n-1$

Computation Time is scalable



- Investigated computational overhead using Pattern Search for parameter optimization
- Customizable number of parameters

Computation time for # of flows	2	4	6
$\frac{\text{new approach}}{\text{standard apprach}}$	1.99	3.85	6.12

Conclusion

- Improved MGF output bound by inserting Lyapunov's inequality
- New approach is always at least as good as standard bound and is minimally invasive
- Can lead to significant output bound improvement
- Allows for a tighter delay bound in a larger network
- Effect increases with number of output bound computations
- Cost of additional parameter to optimize (but can be conveniently scaled)

Conclusion

- Improved MGF output bound by inserting Lyapunov's inequality
- New approach is always at least as good as standard bound and is minimally invasive
- Can lead to significant output bound improvement
- Allows for a tighter delay bound in a larger network
- Effect increases with number of output bound computations
- Cost of additional parameter to optimize (but can be conveniently scaled)

Thank you for your attention!

Q & A

Beck, M. A. (2016).
Advances in Theory and Applicability of Stochastic Network Calculus.
PhD thesis, TU Kaiserslautern.

Ciucu, F. (2007).
Network calculus delay bounds in queueing networks with exact solutions.
In *International Teletraffic Congress (ITC)*, pages 495–506.

Ciucu, F., Poloczek, F., and Schmitt, J. B. (2013).
Sharp bounds in stochastic network calculus.
CoRR, abs/1303.4114.

Nikolaus, P. and Schmitt, J. B.
Improving output bounds in the stochastic network calculus using lyapunov's inequality.
Under review.

Talagrand, M. (1996).
Majorizing measures: the generic chaining.
The Annals of Probability, pages 1049–1103.

Equivalent Relation

Union Bound

For $X_i \geq 0$:

$$\mathrm{P}\left(\max_{i=1,\ldots,n} X_i > a\right) \overset{\text{(UB)}}{\leq} \sum_{i=1}^{n} \mathrm{P}(X_i > a) \overset{\text{(Chernoff)}}{\leq} e^{-\theta a} \sum_{i=1}^{n} \mathrm{E}\left[e^{\theta X_i}\right]$$

$$\Leftrightarrow \quad \mathrm{P}\left(\max_{i=1,\ldots,n} X_i > a\right) \overset{\text{(Chernoff)}}{\leq} e^{-\theta a}\, \mathrm{E}\left[\max_{i=1,\ldots,n} e^{\theta X_i}\right] \leq e^{-\theta a} \sum_{i=1}^{n} \mathrm{E}\left[e^{\theta X_i}\right]$$

- “quasi-Union bound”