Dealing with Dependence in Stochastic Network Calculus – Using Independence as a Bound

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Abstract

Computing probabilistic end-to-end delay bounds is an old, yet still challenging problem. Stochastic network calculus enables closed-form delay bounds for a large class of arrival processes. However, it encounters difficulties in dealing with dependent flows, as standard techniques require to apply Hölder's inequality. In this paper, we present an alternative bounding technique that, under specific conditions, treats them as if flows were independent. We show in two case studies that it provides often better delay bounds while simultaneously significantly improving the computation time.

1 Introduction and Background

Stochastic network calculus (SNC) is a versatile framework to compute stochastic per-flow delay bounds. Developed as a deterministic worst case analysis in the 1990s by Cruz [Cru91a, Cru91b], stochastic extensions of network calculus emerged quickly after. It allows for closed-form solutions for a broad class of arrival and service processes. In [RF11], it has been shown that the SNC branch using moment generating functions [Cha00, Fid06] provides tighter bounds than the approach using envelope functions [CBL06, JL08, Cru96], as it leverages the independence of arrival flows. However, many results limit the end-to-end analysis to pure tandem topologies.

Analyzing more general networks requires to consider also dependent flows at some points in the network, as the sharing of a resource clearly has a mutual impact on the flows' output behavior. Therefore, if we want to obtain the moment generating function (MGF) of aggregated, yet dependent arrival processes $A_1(s,t)$ and $A_2(s,t)$, we typically invoke Hölder's inequality:

$$\mathrm{E}\left[e^{\theta(A_1(s,t)+A_2(s,t))}\right] \leq \mathrm{E}\left[e^{p\theta A_1(s,t)}\right]^{1/p} \cdot \mathrm{E}\left[e^{q\theta A_2(s,t)}\right]^{1/q},$$

where $0 \leq s \leq t$, $\theta > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$. Hölder's inequality is completely oblivious of the actual dependence structure, thus it often leads to very conservative bounds. Furthermore, it places the burden of an additional, nonlinear parameter for each application to optimize.

Dependence of arrivals does not have to be a negative property per se. Taking advantage of the information about the dependence structure to improve upon the bounds has been attempted before. In [DWS15], the functional dependence is estimated using a copula-based approach. In our work, we investigate a simpler alternative, using the independent scenario as an upper bound. To that end, we rely on a characteristic called negative dependence. We explain the main idea with the help of the following, simplistic example.

Consider a single time slot assuming two arrivals, A_1 and A_2, that are multiplexed at one server. Both arrivals send one packet, each independently with probability $p \in (0,1)$, and the server serves one packet but strictly prioritizes A_2. Clearly, their two outputs, D_1 and D_2, are strongly dependent, as an arrival of the prioritized flow forces the other one to wait in the queue. For the joint distribution of the output, we obtain for the departures both being equal to 0, that $\mathrm{P}(D_1 = 0, D_2 = 0) = (1-p)^2$. On the other hand, we compute for the product distribution by a simple conditioning, that $\mathrm{P}(D_1 = 0) \cdot \mathrm{P}(D_2 = 0) = \left((1-p)^2 + p\right)(1-p) > (1-p)^2$. Hence, if we deliberately forego the knowledge about the dependence structure, we only obtain an upper bound, yet, it allows us to consider only the marginal distribution.

2 Definitions and Modeling Assumptions

2.1 SNC Background and Notation

We use the MGF-based SNC in order to calculate per-flow delay bounds. To be precise, we bound the probability that the delay exceeds a given value T. The MGF bound on a probability is established by applying Chernoff's bound [Nel95]

$$\mathrm{P}(X > a) \leq e^{-\theta a}\, \mathrm{E}\left[e^{\theta X}\right], \quad \theta > 0.$$

We define an *arrival flow* by the stochastic process A with discrete time space $\mathbb{N}$ and continuous state space $\mathbb{R}_0^+$ as $A(s,t) := \sum_{i=s+1}^{t} a_i$, with a_i as the traffic increment process in time slot i. Network calculus provides an elegant system-theoretic analysis by employing min-plus algebra.

Definition 1 (Convolution in Min-Plus Algebra [BCOQ92])**.** The min-plus (de-)convolution of real-valued, bivariate functions $x(s,t)$ and $y(s,t)$ is defined as

$$\begin{aligned}(x \otimes y)\,(s,t) &:= \inf_{s \leq i \leq t} \{x(s,i) + y(i,t)\}, \\ (x \oslash y)\,(s,t) &:= \sup_{0 \leq i \leq s} \{x(i,t) - y(i,s)\}.\end{aligned}$$

The characteristics of the service process are captured by the notion of a dynamic S-server.

Definition 2 (Dynamic S-Server [Cha00])**.** Assume a service element has an arrival flow A as its input and the respective output is denoted by D. Let $S(s,t)$, $0 \leq s \leq t$, be a stochastic process that is nonnegative and increasing in t. The service element is a *dynamic S-server* iff for all $t \geq 0$ it holds that

$$D(0,t) \geq (A \otimes S)\,(0,t) = \inf_{0 \leq s \leq t} \{A(0,s) + S(s,t)\}.$$

The analysis is based on a per-flow perspective. That is, we consider a certain flow, the so-called *flow of interest* (foi). Throughout this paper, for the sake of simplicity, we assume the servers' scheduling to be arbitrary multiplexing [SZF08]. That is, if flow f_2 is prioritized over flow f_1, the leftover service at a dynamic S-server for the corresponding arrival A_1 is $S_{\mathrm{l.o.}}(s,t) = [S(s,t) - A_2(s,t)]^+$. Therefore, we require the server to be work-conserving.

Definition 3 (Work-Conserving Server [Cha00][Fid06])**.** For any $t \geq 0$ let $\tau := \sup\{s \in [0,t] : D(0,s) = A(0,s)\}$ be the beginning of the last backlogged period before t. Assume again the service $S(s,t)$, $0 \leq s \leq t$, to be a stochastic process that is nonnegative and increasing in t with $S(\tau,\tau) = 0$. A server is said to be *work-conserving* if for any fixed sample path the server is non-idling and uses the entire available service, i.e., $D(0,t) = D(0,\tau) + S(\tau,t)$.

Definition 4 (Virtual Delay)**.** The *virtual delay* at time $t \geq 0$ is defined as

$$d(t) \coloneqq \inf \{\tau \geq 0 : A(0,t) \leq D(0,t+\tau)\} .$$

It can briefly be described as the time it takes for the cumulated departures to "catch up with" the cumulated arrivals.

Theorem 1 (Output and Delay Bound)**.** *[Cha00][Fid06] Consider an arrival process $A(s,t)$ with dynamic S-server $S(s,t)$.*

The departure process D is upper bounded for any $0 \leq s \leq t$ according to

$$D(s,t) \leq (A \oslash S)\ (s,t). \tag{1}$$

The delay at $t \geq 0$ is upper bounded by

$$d(t) \leq \inf \{\tau \geq 0 : (A \oslash S)\ (t+\tau,t) \leq 0\} .$$

We focus on the analogue of Theorem 1 for moment generating functions:

Theorem 2 (Output and Delay MGF-Bound [Fid06][Bec16])**.** *For the assumptions as in Theorem 1, we obtain:*

The MGF of the departure process D is upper bounded for any $0 \leq s \leq t$ according to

$$\mathrm{E}\left[e^{\theta D(s,t)}\right] \leq \mathrm{E}\left[e^{\theta((A \oslash S)\,(s,t))}\right] . \tag{2}$$

The violation probability of a given stochastic delay bound $T \geq 0$ at time $t \geq 0$ is bounded by

$$\mathrm{P}(d(t) > T) \leq \mathrm{E}\left[e^{\theta((A \oslash S)\,(t+T,t))}\right] . \tag{3}$$

In the following definition, we introduce (σ,ρ)-constraints [Cha00] as they enable us to give bounds under stability conditions.

Definition 5 ((σ,ρ)-Bound [Cha00])**.** An arrival flow is (σ_A,ρ_A)-bounded for some $\theta > 0$, if its MGF exists and for all $0 \leq s \leq t$

$$\mathrm{E}\left[e^{\theta A(s,t)}\right] \leq e^{\theta(\rho_A(\theta)(t-s)+\sigma_A(\theta))}.$$

2.2 Negative Dependence and Acceptable Random Variables

As we explained in the introduction, we would like to bound the joint distribution of two random variables by their respective product distribution. This concept is captured in the 1960s by Lehmann and his notion of negative dependence.

Definition 6 (Negative Dependence [Leh66])**.** A finite family of random variables $\{X_1,\dots,X_n\}$ is said to be *negatively (orthant) dependent (ND)* if the two following inequalities hold:

$$\mathrm{P}(X_1 \leq x_1,\dots,X_n \leq x_n) \leq \prod_{i=1}^{n} \mathrm{P}(X_i \leq x_i)\,,$$

$$\mathrm{P}(X_1 > x_1,\dots,X_n > x_n) \leq \prod_{i=1}^{n} \mathrm{P}(X_i > x_i)\,,$$

for all real numbers $x_1,\dots,x_n$.

The following lemma shows how this characteristic can be used directly in the context of MGFs.

Lemma 1 ([JDP83, Sun11])**.** *If $\{X_n, n \geq 1\}$ is a sequence of ND random variables, then for any $\theta > 0$,*

$$\mathrm{E}\left[e^{\theta \sum_{i=1}^{n} X_i}\right] \leq \prod_{i=1}^{n} \mathrm{E}\left[e^{\theta X_i}\right]. \tag{4}$$

In other words, treating the aggregate of ND random variables as if they were independent yields an upper bound for the respective MGFs. Random variables that suffice Eqn. (4) are called "acceptable" [AKV08], but are studied in an unrelated context.

Showing that two random variables are negatively dependent is a challenging task. Some results exist, e.g., in [JDP83], it has been shown that a permutation distribution, and therefore random sampling without replacement, is ND. In our context, this provides a result for a single time slot. In the following, we confine ourselves to conjecture this property for intervals.

Conjecture 1. *Let two independent flows with according arrival processes A_1 and A_2 traverse a server with finite capacity. Further, both arrival processes have iid increments. Then, we assume their respective output processes $D_1(s,t)$ and $D_2(s,t)$ to be ND for all intervals with $0 \leq s \leq t$.*

We do not have a proof but it held in all our experiments using 10^6 samples to estimate the joint and product (C)CDFs, respectively: For two flows with exponentially distributed packet sizes at one server, we tried over 3800 different combinations of intervals, x_1, x_2, (as in the CDF) and utilizations (between 0.4 and 0.9).

The focus on the same interval for both process is important, as the following, admittedly simplifying, argument suggests: Assume the high priority (HP) flow to send a lot of packets consecutively, i.e., the low priority (LP) flow has no output in this period and queues all its packets. Then, it is more likely for the LP flow to have outputs when the HP flow stops sending, as it is more likely for it to have queued packets.

3 Independence as a Bound

In this section, we investigate two cases studies to show in which part of the analysis we exploit the negative dependence.

In the following, we call the flow f_1, whose delay we stochastically upper bound, flow of interest (foi). All arrival processes A_i are assumed to be discrete time and to have iid increments and all servers S_j are work-conserving and provide a constant rate $c_j \geq 0$. To simplify notation, we denote by $D_i^{(j)}$ the output of flow i at server j.

3.1 Diamond Network

In this case study, we consider the topology in Fig. 1. Assume the foi to have the lowest priority and f_3 to have the highest priority. By SNC literature [CBL06, Fid06], the service provided for the flow of interest, also known as the network service curve, can be described by

$$S_{\text{net}} = \left[S_1 - \left(\left(\left(A_2 \oslash [S_4 - A_3]^+\right) \oslash S_2\right) + \left((A_3 \oslash S_4) \oslash S_3\right)\right)\right]^+.$$

Since Conjecture 1 is made on output processes, we postpone the application of the output bound in Eqn. (1) by keeping the exact output at first. That is, we start with

$$S_{\text{net}} = \left[S_1 - \left(D_2^{(2)} + D_3^{(3)}\right)\right]^+,$$

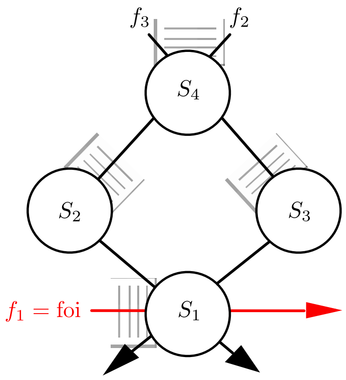


Fig. 1: Diamond network.

use then the conjecture to bound the MGF of the aggregate by their product (Eqn. (4)), and apply the output bound in a final step.

The probability that the delay process $d(t)$ exceeds a value $T \geq 0$ is upper bounded by

$$\begin{aligned}
&\mathrm{P}(d(t) > T) \\
&\overset{(3)}{\leq} \mathrm{E}\left[e^{\theta (A_1 \oslash S_{\mathrm{net}})(t+T,t)}\right] \\
&\leq \sum_{\tau_1=0}^{t} \mathrm{E}\left[e^{\theta (A_1(\tau_1,t) - S_{\mathrm{net}}(\tau_1,t+T))}\right] \\
&= \sum_{\tau_1=0}^{t} \mathrm{E}\left[e^{\theta A_1(\tau_1,t)}\right] \mathrm{E}\left[e^{-\theta \left[S_1 - \left(D_2^{(2)} + D_3^{(3)}\right)\right]^+ (\tau_1,t+T)}\right] \\
&\leq \sum_{\tau_1=0}^{t} \mathrm{E}\left[e^{\theta A_1(\tau_1,t)}\right] e^{-\theta c_1 (t+T-\tau_1)} \mathrm{E}\left[e^{\theta \left(D_2^{(2)} + D_3^{(3)}\right)(\tau_1,t+T)}\right], \qquad (5)
\end{aligned}$$

where we used Theorem 2 in the first inequality and the Union bound in the line below. Since the flows f_2 and f_3 share the server S_4, their according output processes $D_2^{(4)}$ and $D_3^{(4)}$ are dependent and, as a consequence, $D_2^{(2)}$ and $D_3^{(3)}$, as well. However, by the conjecture above, we assume that the resource sharing at S_4 indicates that the dependence structure on $[\tau_1, t+T]$ is of a negative nature which, in turn, is the reason why we upper bound them as if they were independent.

This can be interpreted as if we analyzed a new system, where the server S_4 would be split into two servers. That is, one provides the same service as the original (for the high priority flow f_3), and the other provides the leftover service $[S_4' - A_3']^+$, where S_4' has the same service rate as S_4 and A_3' is a new arrival process, but with the same distribution as A_3.

Hence, the second factor is upper bounded by

$$\begin{aligned}
\mathrm{E}\left[e^{\theta \left(D_2^{(2)} + D_3^{(3)}\right)(\tau_1,t+T)}\right] &\leq \mathrm{E}\left[e^{\theta D_2^{(2)}(\tau_1,t+T)}\right] \mathrm{E}\left[e^{\theta D_3^{(3)}(\tau_1,t+T)}\right] \\
&\leq \mathrm{E}\left[e^{\theta \left(\left(A_2 \oslash [S_4 - A_3]^+\right) \oslash S_2\right)(\tau_1,t+T)}\right] \mathrm{E}\left[e^{\theta ((A_3 \oslash S_4) \oslash S_3)(\tau_1,t+T)}\right].
\end{aligned}$$

This gives us

$$
\begin{aligned}
&\mathrm{P}(d(t) > T) \\
&\le \sum_{\tau_1=0}^{t} \mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\, e^{-\theta c_1(t+T-\tau_1)}\, \mathrm{E}\Big[e^{\theta\left(\left(A_2 \oslash [S_4 - A_3]^+\right)\oslash S_2\right)(\tau_1,t+T)}\Big]\, \mathrm{E}\Big[e^{\theta((A_3 \oslash S_4)\oslash S_3)(\tau_1,t+T)}\Big] \\
&\le \sum_{\tau_1=0}^{t} \mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\, e^{-\theta c_1(t+T-\tau_1)} \left\{ \sum_{\tau_2=0}^{\tau_1} \mathrm{E}\Big[e^{\theta\left(A_2 \oslash [S_4-A_3]^+\right)(\tau_2,t+T)}\Big]\, \mathrm{E}\Big[e^{-\theta S_2(\tau_2,\tau_1)}\Big] \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \mathrm{E}\Big[e^{\theta (A_3 \oslash S_4)(\tau_2,t+T)}\Big]\, \mathrm{E}\Big[e^{-\theta S_3(\tau_2,\tau_1)}\Big] \right\} \\
&\le \sum_{\tau_1=0}^{t} \mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\, e^{-\theta c_1(t+T-\tau_1)} \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} \mathrm{E}\Big[e^{\theta A_2(\tau_3,t+T)}\Big]\, \mathrm{E}\Big[e^{\theta A_3(\tau_3,\tau_2)}\Big]\, \mathrm{E}\Big[e^{-\theta S_4(\tau_3,\tau_2)}\Big] \right\} \mathrm{E}\Big[e^{-\theta S_2(\tau_2,\tau_1)}\Big] \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} \mathrm{E}\Big[e^{\theta A_3(\tau_3,t+T)}\Big]\, \mathrm{E}\Big[e^{-\theta S_4(\tau_3,,\tau_2)}\Big] \right\} \mathrm{E}\Big[e^{-\theta S_3(\tau_2,\tau_1)}\Big] \right\},
\end{aligned}
$$

after applying the Union bound for each usage of the deconvolution. Further assuming all A_i to be (σ_A, ρ_A)-bounded [Cha00] yields a closed-form for the delay bound under the stability condition

$$
\begin{aligned}
\rho_{A_1}(\theta) + \rho_{A_2}(\theta) + \rho_{A_3}(\theta) <& c_1, \\
\rho_{A_2}(\theta) <& c_2, \\
\rho_{A_3}(\theta) <& c_3, \\
\rho_{A_2}(\theta) + \rho_{A_3}(\theta) <& c_4 :
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{P}(d(t) > T) \\
&\le \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)(t-\tau_1)+\sigma_1(\theta)\right)} e^{-\theta c_1(t+T-\tau_1)} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} e^{\theta\left(\rho_{A_2}(\theta)(t+T-\tau_3)+\sigma_{A_2}(\theta)\right)} e^{\theta\left(\rho_{A_3}(\theta)(\tau_2-\tau_3)+\sigma_{A_3}(\theta)\right)} e^{-\theta c_4(\tau_2-\tau_3)} \right\} e^{-\theta c_2(\tau_1-\tau_2)} \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} e^{\theta\left(\rho_{A_3}(\theta)(t+T-\tau_3)+\sigma_{A_3}(\theta)\right)} e^{-\theta c_4(\tau_2-\tau_3)} \right\} e^{-\theta c_3(\tau_1-\tau_2)} \right\} \\
&= e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)T+\sigma_1(\theta)+\sigma_{A_2}(\theta)+2\sigma_{A_3}(\theta)\right)} \\
&\quad \cdot \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)-c_1\right)(t-\tau_1)} \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} e^{\theta\rho_{A_2}(\theta)(t-\tau_3)} e^{\theta\left(\rho_{A_3}(\theta)-c_4\right)(\tau_2-\tau_3)} \right\} e^{-\theta c_2(\tau_1-\tau_2)} \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \left\{ \sum_{\tau_3=0}^{\tau_2} e^{\theta\rho_{A_3}(\theta)(t-\tau_3)} e^{-\theta c_4(\tau_2-\tau_3)} \right\} e^{-\theta c_3(\tau_1-\tau_2)} \right\} \\
&\le e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)T+\sigma_1(\theta)+\sigma_{A_2}(\theta)+2\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)-c_1\right)(t-\tau_1)} \left\{ \sum_{\tau_2=0}^{\tau_1} \frac{e^{\theta\rho_{A_2}(\theta)(t-\tau_2)}}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_4\right)}} e^{-\theta c_2(\tau_1-\tau_2)} \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \frac{e^{\theta\rho_{A_3}(\theta)(t-\tau_2)}}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_4\right)}} e^{-\theta c_3(\tau_1-\tau_2)} \right\}
\end{aligned}
$$

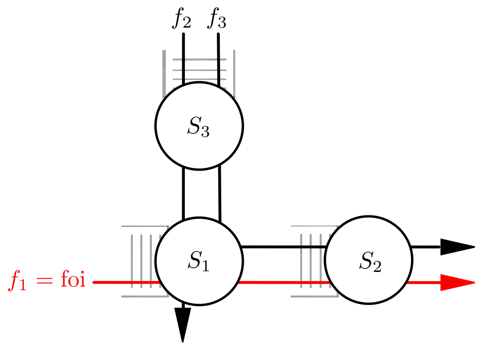


Fig. 2: The $\mathbb{L}$.

$$
\begin{aligned}
&\le e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)T+\sigma_1(\theta)+\sigma_{A_2}(\theta)+2\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)(t-\tau_1)} \left\{ \sum_{\tau_2=0}^{\tau_1} \frac{e^{\theta\left(\rho_{A_2}(\theta)-c_2\right)(\tau_1-\tau_2)}}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_4\right)}} \right\} \\
&\quad \cdot \left\{ \sum_{\tau_2=0}^{\tau_1} \frac{e^{\theta\left(\rho_{A_3}(\theta)-c_3\right)(\tau_1-\tau_2)}}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_4\right)}} \right\} \\
&\le e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)T+\sigma_1(\theta)+\sigma_{A_2}(\theta)+2\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} \frac{e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)(t-\tau_1)}}{1-e^{\theta\left(\rho_{A_2}(\theta)-c_2\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_3\right)}} \\
&\quad \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_4\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_4\right)}} \\
&\le \frac{e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)T+\sigma_1(\theta)+\sigma_{A_2}(\theta)+2\sigma_{A_3}(\theta)\right)}}{1-e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_1\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)-c_2\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_3\right)}} \\
&\quad \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_4\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_3}(\theta)-c_4\right)}}.
\end{aligned}
$$

In contrast, standard techniques proceed at Eqn. (5) by applying the output Bound Eqn. (1) immediately and continue with Hölder's inequality to deal with the dependence.

3.2 The $\mathbb{L}$

In this case study, we analyze the topology in Fig. 2. The foi has the lowest priority and f_2 the highest. Similarly to Subsection 3.1, we assume the outputs processes of f_2 and f_3 to be ND, based on Conjecture 1. Here, the network service curve is

$$S_{\text{net}} = \left[\left([S_1 - (A_2 \oslash S_3)]^+ \otimes S_2\right) - \left(A_3 \oslash [S_3 - A_2]^+\right)\right]^+ .$$

Again, we postpone the output bounding and start with

$$S_{\text{net}} = \left[\left(\left[S_1 - D_2^{(3)}\right]^+ \otimes S_2\right) - D_3^{(3)}\right]^+ .$$

The crucial difference is that, in order to obtain the delay bound for the foi, the so-called min-plus convolution has to be applied to the service processes of S_1 and S_2 forcing us to analyze the output processes at different intervals:

$$
\begin{aligned}
&\mathrm{P}(d(t)>T)\\
&\overset{(3)}{\leq} \mathrm{E}\Big[e^{\theta(A_1\oslash S_{\mathrm{net}})\,(t+T,t)}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta(A_1(\tau_1,t)-S_{\mathrm{net}}(\tau_1,t+T))}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\,\mathrm{E}\left[e^{-\theta\left[\left(\left[S_1-D_2^{(3)}\right]^+\otimes S_2\right)-D_3^{(3)}\right]^+(\tau_1,t+T)}\right]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\,\mathrm{E}\left[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{-\theta\left(\left[S_1-D_2^{(3)}\right]^+\otimes S_2\right)(\tau_1,t+T)}\right]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{-\theta\left[S_1-D_2^{(3)}\right]^+(\tau_1,\tau_2)}e^{-\theta S_2(\tau_2,t+T)}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{\theta D_2^{(3)}(\tau_1,\tau_2)}e^{-\theta S_1(\tau_1,\tau_2)}e^{-\theta S_2(\tau_2,t+T)}\Big]\\
&= \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\,\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{\theta D_2^{(3)}(\tau_1,\tau_2)}\Big],
\end{aligned}
$$

where we used the Union bound for each application of the convolution / deconvolution. This scenario is not covered by Conjecture 1 (see also the discussion at the end of Subsection 2.2). Our workaround is to leverage the monotonicity of $D_2^{(3)}$:

$$
\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{\theta D_2^{(3)}(\tau_1,\tau_2)}\Big]\leq \mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{\theta D_2^{(3)}(\tau_1,t+T)}\Big].
$$

We then continue with

$$
\begin{aligned}
&\mathrm{P}(d(t)>T)\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\,\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}e^{\theta D_2^{(3)}(\tau_1,t+T)}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\,\mathrm{E}\Big[e^{\theta D_3^{(3)}(\tau_1,t+T)}\Big]\,\mathrm{E}\Big[e^{\theta D_2^{(3)}(\tau_1,t+T)}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\,\mathrm{E}\Big[e^{\theta(A_2\oslash S_3)(\tau_1,t+T)}\Big]\,\mathrm{E}\Big[e^{\theta\left(A_3\oslash[S_3-A_2]^+\right)(\tau_1,t+T)}\Big]\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\left\{\sum_{\tau_3=0}^{\tau_1}\mathrm{E}\Big[e^{\theta A_2(\tau_3,t+T)}\Big]\,e^{-\theta c_3(\tau_1-\tau_3)}\right\}\\
&\quad\cdot\left\{\sum_{\tau_3=0}^{\tau_1}\mathrm{E}\Big[e^{\theta A_3(\tau_3,t+T)}\Big]\,\mathrm{E}\Big[e^{-\theta[S_3-A_2]^+(\tau_3,\tau_1)}\Big]\right\}\\
&\leq \sum_{\tau_1=0}^{t}\mathrm{E}\Big[e^{\theta A_1(\tau_1,t)}\Big]\sum_{\tau_2=\tau_1}^{t+T}e^{-\theta c_1\cdot(\tau_2-\tau_1)}e^{-\theta c_2\cdot(t+T-\tau_2)}\left\{\sum_{\tau_3=0}^{\tau_1}\mathrm{E}\Big[e^{\theta A_2(\tau_3,t+T)}\Big]\,e^{-\theta c_3(\tau_1-\tau_3)}\right\}
\end{aligned}
$$

$$\cdot \left\{ \sum_{\tau_3=0}^{\tau_1} \mathrm{E}\Big[e^{\theta A_3(\tau_3,t+T)}\Big]\,\mathrm{E}\Big[e^{\theta A_2(\tau_3,\tau_1)}\Big]\,e^{-\theta c_3(\tau_1-\tau_3)} \right\}.$$

If we again assume all A_i to be (σ_A, ρ_A)-bounded, we obtain for

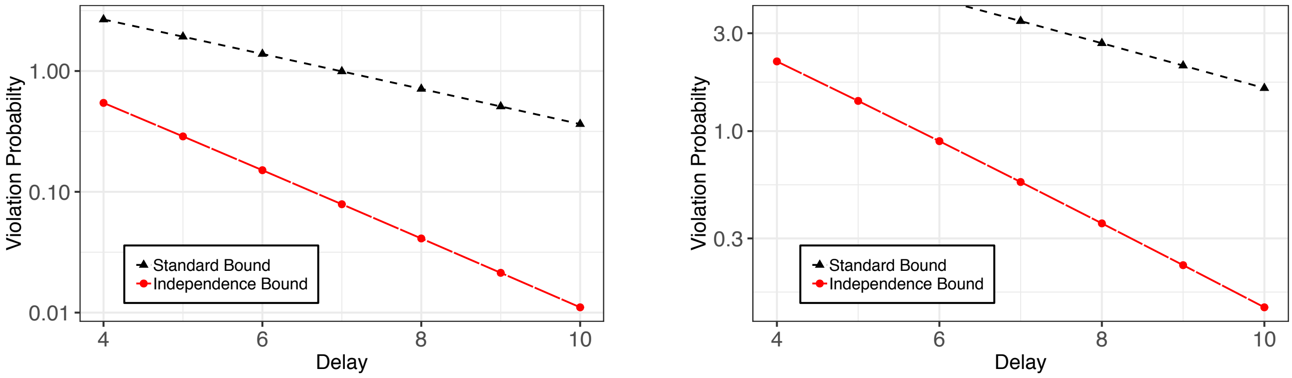
$$\begin{aligned}\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta) &< \min\{c_1, c_2\},\\ \rho_{A_2}(\theta)+\rho_{A_3}(\theta) &< c_3,\end{aligned}$$

and $c_1 \neq c_2$:

$$\begin{aligned}
&\mathrm{P}(d(t)>T)\\
&\leq \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)(t-\tau_1)+\sigma_{A_1}(\theta)\right)} \sum_{\tau_2=\tau_1}^{t+T} e^{-\theta c_1\cdot(\tau_2-\tau_1)} e^{-\theta c_2\cdot(t+T-\tau_2)} \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)(t+T-\tau_3)+\sigma_{A_2}(\theta)\right)} e^{-\theta c_3(\tau_1-\tau_3)} \right\}\\
&\quad\cdot \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)(\tau_1-\tau_3)+\sigma_{A_2}(\theta)\right)} e^{\theta\left(\rho_{A_3}(\theta)(t+T-\tau_3)+\sigma_{A_3}(\theta)\right)} e^{-\theta c_3(\tau_1-\tau_3)} \right\}\\
&\leq e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)\cdot T+\sigma_{A_1}(\theta)+2\sigma_{A_2}(\theta)+\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} e^{\theta\rho_{A_1}(\theta)(t-\tau_1)} \sum_{\tau_2=\tau_1}^{t+T} e^{-\theta c_1\cdot(\tau_2-\tau_1)} e^{-\theta c_2\cdot(t+T-\tau_2)}\\
&\quad\cdot \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)(t-\tau_3)\right)} e^{-\theta c_3(\tau_1-\tau_3)} \right\} \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)-c_3\right)(\tau_1-\tau_3)} e^{\theta\rho_{A_3}(\theta)(t-\tau_3)} \right\}\\
&\leq e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)\cdot T+\sigma_{A_1}(\theta)+2\sigma_{A_2}(\theta)+\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)(t-\tau_1)} \sum_{\tau_2=\tau_1}^{t+T} e^{-\theta c_1\cdot(\tau_2-\tau_1)} e^{-\theta c_2\cdot(t+T-\tau_2)}\\
&\quad\cdot \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)-c_3\right)(\tau_1-\tau_3)} \right\} \left\{ \sum_{\tau_3=0}^{\tau_1} e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_3\right)(\tau_1-\tau_3)} \right\}\\
&\leq e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)\cdot T+\sigma_{A_1}(\theta)+2\sigma_{A_2}(\theta)+\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} \frac{e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)(t-\tau_1)}}{1-e^{\theta\left(\rho_{A_2}(\theta)-c_3\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_3\right)}}\\
&\quad\cdot \sum_{\tau_2=\tau_1}^{t+T} e^{-\theta c_1\cdot(\tau_2-\tau_1)} e^{-\theta c_2\cdot(t+T-\tau_2)}\\
&\leq e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)\cdot T+\sigma_{A_1}(\theta)+2\sigma_{A_2}(\theta)+\sigma_{A_3}(\theta)\right)} \sum_{\tau_1=0}^{t} \frac{e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)\right)(t-\tau_1)}}{1-e^{\theta\left(\rho_{A_2}(\theta)-c_3\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_3\right)}}\\
&\quad\cdot \sum_{\tau_2=\tau_1}^{t+T} e^{-\theta c_1\cdot(\tau_2-\tau_1)} e^{-\theta c_2\cdot(t+T-\tau_2)}\\
&\leq \frac{e^{\theta\left(\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-\min\{c_1,c_2\}\right)\cdot T+\sigma_{A_1}(\theta)+2\sigma_{A_2}(\theta)+\sigma_{A_3}(\theta)\right)}}{1-e^{\theta\left(\rho_{A_1}(\theta)+\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-\min\{c_1,c_2\}\right)}}\\
&\quad\cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)-c_3\right)}} \cdot \frac{1}{1-e^{\theta\left(\rho_{A_2}(\theta)+\rho_{A_3}(\theta)-c_3\right)}} \cdot \frac{1}{1-e^{-\theta|c_1-c_2|}}.
\end{aligned}$$

4 Numerical Evaluation

We present the results of a numerical evaluation for both case studies. We ran 10^4 Monte-Carlo simulations to sample the parameters for different server rates and packet sizes, the latter sampled from an exponential



(a) Delay bound diamond network.

(b) Delay bound in the $\mathbb{L}$.

Fig. 3: Case study delay bounds.

distribution. The scenarios are then filtered to ensure a utilization $\in [0.5, 1)$.

4.1 Quality of the Bounds

Diamond Network: This topology, after above mentioned filtering, yields 485 remaining scenarios, of which 371 are improved. The fact that not all are improved despite the avoidance of Hölder's inequality can be explained as follows: In the analysis, the Union bound is applied after Hölder's inequality. The exponentiation before the summing followed by a square root can have a reducing effect. A similar observation has been exploited in SNC literature before [**?**].

We also measured the extent of the improvement by computing the ratio of the delay violation probability of the standard approach over the "independence bound". Clearly, values above 1 are desirable. Here, we obtain a median improvement of 6.04. In Fig. 3a, we depict the delay bounds for a specific parameter set.

The $\mathbb{L}$: Here, we expect a weaker performance, as our approach using independence requires the additional step of extending the interval of one output process. The numerical results confirm this expectation: Out of the 729 scenarios, only half of them (384) yield a performance gain. The median of the improvement ratio confirms this, being relatively close to 1 (1.27). Again, we show the delay bounds for a fixed parameter set (Fig. 3b).

4.2 Computation Run Time

Our proposed approach not only often substantially improves the bounds but it also has a much lower computation complexity than the standard approach. The reason is that the latter relies on an additional Hölder parameter. The optimizations are conducted using a grid search followed by a downhill simplex algorithm. The improvements ratios are in the median 337.5 (1.62 sec compared to 0.0048 sec) for the diamond scenario and 458.1 for the $\mathbb{L}$ (1.42 sec compared to 0.0031 sec).

5 Discussion

In this paper, we found some interesting results indicating that by using independence as a bound, one can often times improve the delay bound while also speeding up the run time significantly. Obviously, the crucial next step is to find scenarios in which the conjecture can be proved rigorously. Furthermore, more scenarios can be analyzed in which the negative dependence can be exploited. In particular, this includes large-scale experiments that require many invocations of Hölder's inequality.

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